Convex integer minimization in fixed dimension

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Abstract

We show that minimizing a convex function over the integer points of a bounded convex set is polynomial in fixed dimension.

1 Introduction.

One of the most important complexity results in integer programming states that minimizing a linear function over the integer points in a polyhedron is solvable in polynomial time provided that the number of integer variables is a constant. This landmark result due to Lenstra [10] has been generalized by Barvinok [2]: he shows that one can even count the number of integer points in polytopes in fixed dimension. More recent extensions of Lenstra's algorithm apply to integer optimization problems associated with semi-algebraic sets and described by quasi-convex polynomials. The first polynomial time algorithm for minimizing a quasi-convex polynomial over such sets in fixed dimension is due to Khachiyan and Porkolab [9]. Recent improvements of the complexity bound are given by Heinz [6] and by Hildebrand and Köppe [7].

In this paper, we drop the assumption that the functions describing the input to our problem are polynomials. Instead, we aim at minimizing a general convex or quasi-convex function over the integer points in a bounded convex set in fixed dimension. The bounded convex set is defined by convex or quasi-convex functions as well. We assume that all the functions are encoded by means of evaluation oracles: queried on a rational point, the evaluation oracles return the function values that the point attains. We assume that three further oracles are given, namely a continuous feasibility oracle, a separating hyperplane oracle and a linear integer optimization oracle. In order to realize them one needs additional assumptions on the functions.

It is well known that Lenstra's algorithm can in principle be applied to any class of convex sets $\mathcal{C}$ in $\mathbb{R}^n$ when n is a constant provided that we can determine an ellipsoidal approximation for every member in $\mathcal{C}$ efficiently. By an *ellipsoidal approximation* of a convex set we mean an ellipsoid E contained in the convex set such that a properly scaled version of E contains the convex set. (Typically, the scaling factor is $\mathcal{O}(n)$). The construction of such an ellipsoidal approximation is often performed by designing a *shallow cut separation oracle* (see, for instance, [4, Section 3.3]). To the best of our knowledge it is not known how to construct ellipsoidal approximations for general convex sets in polynomial time even when the number of variables is fixed. This explains why general convex integer minimization problems with a fixed number of variables have not yet been extensively studied.

We design a novel polynomial time algorithm for general convex integer minimization problems in fixed dimension that avoids going through the construction of ellipsoidal approximations. Instead we develop a cone-shrinking algorithm that from iteration to iteration produces smaller and smaller cones containing the convex set under consideration until we can reduce the original question to a series of similar problems in smaller dimensions.

Our assumptions are as follows. Let $f_0, \ldots, f_m : \mathbb{R}^n \mapsto \mathbb{R}$ be quasi-convex functions, i.e. for every $\alpha \in \mathbb{R}$ the level set $\{x \in \mathbb{R}^n \mid f_i(x) \leq \alpha\}$ is convex. Note that a convex function is also quasi-convex. For a given $\varepsilon \in \mathbb{R}_{>0}$, we define

$$K_0 := \{x \in \mathbb{R}^n \mid f_i(x) \leq 0 \text{ for all } i = 0, \ldots, m\},$$
$$K_\varepsilon := \{x \in \mathbb{R}^n \mid f_i(x) \leq \varepsilon \text{ for all } i = 0, \ldots, m\}.$$

Moreover, let $B, \Delta, M \in \mathbb{N}$ be given numbers. We assume that $K_0,\ K_\varepsilon \subset [-B, B]^n$ and that $|f_i(x)| \leq M$ for all $x \in [-B, B]^n$ and all $i = 0, ..., m$. For a point $x \in \mathbb{Q}^n$ the precision of x is the smallest integer $q \in \mathbb{N}$ such

that x has a representation $x = (\frac{p_1}{q}, \ldots, \frac{p_n}{q})$, where $p_j \in \mathbb{Z}$ for all $j = 1, \ldots, n$. We are interested in rational points with a precision of at most Δ. We assume to have available three oracles.

Δ-Feasibility Oracle. Given a polytope P in $[-B, B]^n$, the oracle returns a point $x \in P \cap K_{\frac{\varepsilon}{2}}$ with precision at most Δ, or certifies that $P \cap K_0$ does not contain a point with precision at most Δ.

Separating Hyperplane Oracle. Given an affine space A and a point $a \in A$, the oracle returns either that $a \in K_\varepsilon \cap A$ or it returns a vector $c \in \mathbb{Q}^n$ with $\|c\|_\infty = 1$ such that $c^\mathsf{T} x \leq c^\mathsf{T} a$ for every $x \in K_{\frac{\varepsilon}{2}} \cap A$.

Linear Integer Optimization Oracle. Given a polytope P and a linear objective function, the oracle returns a point in $P \cap \mathbb{Z}^n$ with minimum objective function value, or certifies that $P \cap \mathbb{Z}^n$ is empty.

The polytope P that is part of the input in the Δ-Feasibility Oracle will always be defined as the intersection of the box $[-B, B]^n$ with an affine space. Polynomial time algorithms for realizing a Δ-Feasibility Oracle can be found in [3] and [11].

To the best of our knowledge there is no efficient algorithm for realizing a Separating Hyperplane Oracle in general. Rather, concrete realizations depend on properties of the functions f_i, $i = 0, \ldots, m$. One particularly relevant case in which the Separating Hyperplane Oracle can be emulated is as follows. Let us assume that the functions $f_0, \ldots, f_m$ are convex. Moreover, let us assume that, for every $x \in [-B, B]^n$ and for every $i \in \{0, \ldots, m\}$, a subgradient of $\partial f_i(x)$ is known. Suppose now that an affine space A and a point $a \in A$ are given. The question is to decide whether $a \in K_\varepsilon \cap A$, or – if not – to find a hyperplane that separates a from $K_{\frac{\varepsilon}{2}} \cap A$. We start by checking whether $a \in K_\varepsilon \cap A$. This can be done by simply substituting a into the functions f_i, $i = 0, \ldots, m$. Let us assume that $a \notin K_\varepsilon \cap A$. Then there exists one of the functions f_i, say f_0, such that $f_0(a) > \varepsilon > \frac{\varepsilon}{2}$. We take an element $g \in \partial f_0(a)$. Note that g is the normal vector of a tangent hyperplane H of the epigraph of f_0 at the point $(a, f_0(a))$. Next we shift H such that it contains the point a. Let the resulting hyperplane be H'. Then $H' \cap A$ is a separating hyperplane.

For a realization of the Linear Integer Optimization Oracle we refer again to the paper of Lenstra [10].

We emphasize that the parameter ε does not affect the number of iterations of our cone-shrinking algorithm. In fact, from now on we assume that ε is fixed. Of course, it plays a role in the realizations of our oracles. Our main contribution is stated in the theorem below.

Theorem 1.1. *Let $f_0, \ldots, f_m$, K_0, K_ε, B, and Δ be as above. In polynomial time in $\log(B)$ and $\log(\Delta)$ either one can find a point $z \in K_\varepsilon \cap \mathbb{Z}^n$, or show that $K_0 \cap \mathbb{Z}^n = \emptyset$.*

Theorem 1.1 allows us to minimize any quasi-convex function f_0 over the integer points of a convex set $L := \{x \in [-B, B]^n \mid f_i(x) \leq 0, \text{ for all } i = 1, \ldots, m\}$ in polynomial time, when n is fixed. An approximate solution of the problem $\min\{f_0(x) \mid x \in L \cap \mathbb{Z}^n\}$ can be computed by binary search in the interval $[-M, M]$. This follows since, for any $\gamma \in [-M, M]$, Theorem 1.1 can be applied to the set $\{x \in [-B, B]^n \mid f_0(x) - \gamma \leq 0,\ f_i(x) \leq 0, \text{ for all } i = 1, \ldots, m\}$ instead of K_0. We thus derive the following corollary.

Corollary 1.2. *Let $f_0, \ldots, f_m$, B, Δ, and M be as above. In polynomial time in $\log(B)$, $\log(\Delta)$ and $\log(M)$ either one can find a point $z \in \mathbb{Z}^n$ such that $f_0(z) \leq \min\{f_0(x) \mid x \in \mathbb{Z}^n \text{ and } f_i(x) \leq \varepsilon \text{ for all } i = 1, \ldots, m\} + \varepsilon$, or show that the problem $\min\{f_0(x) \mid x \in \mathbb{Z}^n \text{ and } f_i(x) \leq 0 \text{ for all } i = 1, \ldots, m\}$ is infeasible.*

In the next section, we introduce the notation and we prove statements that are needed to show Theorem 1.1. Section 3 contains the proof of Theorem 1.1. Section 4 describes a straightforward generalization of our algorithm to the mixed integer case.

2 Auxiliary lemmata.

For a set $S \subset \mathbb{R}^n$ we denote by $\mathrm{aff}(S)$ the affine hull of S, by $\mathrm{conv}(S)$ the convex hull of S, by $\mathrm{int}(S)$ the interior of S, by $\dim(S)$ the dimension of the smallest affine space containing S, and by $\mathrm{vol}_j(S)$, $j = 1, \ldots, n$, the Lebesgue measure of S with respect to a j-dimensional affine subspace containing it. We omit the subscript and simply write $\mathrm{vol}(S)$ whenever the dimension is clear from the context. For two sets $S, T \subset \mathbb{R}^n$

we denote by $S+T := \{s+t : s \in S,\ t \in T\}$ the Minkowski sum of S and T. When S is bounded, the set $S-S := \{x-y : x, y \in S\}$ is called the difference body of S. If $M \in \mathbb{R}^{n\times n}$ is a matrix, then $\det(M)$ denotes the determinant of M.

In the remainder of this section we present five lemmata. Lemmata 2.2 and 2.5 are needed to prove Lemma 2.6. Lemmata 2.1, 2.4, and 2.6 are used in the next section to prove Theorem 1.1. The following lemma states that the convex hull of the integer points of an n-dimensional closed convex set is lower-dimensional whenever its volume is sufficiently small.

Lemma 2.1. *Let $K \subset \mathbb{R}^n$ be a closed convex set such that $\operatorname{vol}(K) < \frac{1}{n!}$. Then $\dim(\operatorname{conv}(K \cap \mathbb{Z}^n)) \leq n-1$.*

Proof. For the purpose of deriving a contradiction assume that there exist $n+1$ affinely independent points $v_0, \ldots, v_n \in K \cap \mathbb{Z}^n$. Then $\operatorname{vol}(\operatorname{conv}(\{v_0, \ldots, v_n\})) = \frac{1}{n!}|\det(v_1 - v_0, \ldots, v_n - v_0)| \geq \frac{1}{n!}$. □

In the subsequent lemma we define for every n-dimensional closed convex set K a corresponding set $\bar{K}$ that is contained in K. The set $\bar{K}$ has the property that the Minkowski sum of $\bar{K}$ and a certain scaling of the difference body of K is a subset of K. Then, Lemma 2.2 gives an outer approximation of K and an inner approximation of $\bar{K}$ in terms of a certain ellipsoid. Consequently, this ellipsoid can be used to approximate $K \setminus \bar{K}$. Furthermore, we always have that $\bar{K} \neq \emptyset$.

Lemma 2.2. *Let $K \subset \mathbb{R}^n$ be an n-dimensional closed convex set, and let*

$$\bar{K} := \left\{ x \in \mathbb{R}^n \;\middle|\; x + \frac{1}{4n}(K-K) \subset K \right\}.$$

Then there exists an ellipsoid $E \subset \mathbb{R}^n$ and a point $c \in \bar{K}$ such that $c + \frac{1}{2}E \subset \bar{K}$ and $K \subset c + nE$.

Proof. By John's characterization of inscribed ellipsoids of maximal volume (see John [8] and Ball [1]), there exists an ellipsoid E centered at the origin, and a point c such that $c + E \subset K \subset c + nE$. By the definition of the difference body $K-K$, it follows that $2E = E - E \subset K - K \subset nE - nE = 2nE$. This implies $\frac{1}{4n}(K-K) \subset \frac{1}{2}E$ and thus $\frac{1}{2}E + \frac{1}{4n}(K-K) \subset E \subset K - c$. Hence, $(c + \frac{1}{2}E) + \frac{1}{4n}(K-K) \subset K$. This implies that $c + \frac{1}{2}E \subset \bar{K}$. In particular, $c \in \bar{K}$. □

Remark 2.3. *If $K \subset \mathbb{R}^n$ is a polytope, then the set $\bar{K}$ as defined in the previous lemma can be computed explicitly. For that, let $K = \{x \in \mathbb{R}^n \mid a_i^\intercal x \leq b_i,$ for $i = 1, \ldots, m\}$ be represented by facet-defining inequalities. Then, for all $i = 1, \ldots, m$, we define $\rho_i := b_i - \min\{a_i^\intercal x \mid x \in K\}$, i.e. the width of K with respect to a_i. Since for the difference body $K-K$ it holds that $\max\{a_i^\intercal x \mid x \in K-K\} - \min\{a_i^\intercal x \mid x \in K-K\} = 2\rho_i$ for all i, it follows that $\bar{K} = \{x \in \mathbb{R}^n \mid a_i^\intercal x \leq b_i - \frac{1}{4n}\rho_i,$ for $i = 1, \ldots, m\}$.*

For the two sets K and $\bar{K}$ defined in Lemma 2.2, we prove next that when intersecting K with a half-space containing a point of $\bar{K}$ on its boundary, the volume of this intersection is guaranteed to decrease by a constant factor that is only dependent on the dimension.

Lemma 2.4. *Let $K \subset \mathbb{R}^n$ be an n-dimensional closed convex set, and let $\bar{K}$ be defined as in Lemma 2.2. Furthermore, let $x^\star \in \bar{K}$ and let H^+ be a half-space with $x^\star$ on its boundary. Then*

$$\operatorname{vol}(K \cap H^+) \leq \left(1 - \frac{1}{n^n 2^{n+1}}\right) \operatorname{vol}(K).$$

Proof. The Brunn-Minkowski inequality (see, for instance, Gruber [5, Theorem 8.5]) states that $2^n \operatorname{vol}(K) \leq \operatorname{vol}(K-K)$. In addition, we have $x^\star + \frac{1}{4n}(K-K) \subset K$. Furthermore, due to the central symmetry of the difference body $K-K$, we have

$$\operatorname{vol}\left(\left(x^\star + \frac{1}{4n}(K-K)\right) \cap H^+\right) = \frac{1}{2}\operatorname{vol}\left(\frac{1}{4n}(K-K)\right).$$

Hence,

$$\operatorname{vol}(K \cap H^+) \leq \operatorname{vol}(K) - \frac{1}{2}\operatorname{vol}\left(\frac{1}{4n}(K-K)\right) \leq \left(1 - \frac{1}{n^n 2^{n+1}}\right)\operatorname{vol}(K).$$

□

The following lemma is one of the key ingredients of our proof of Theorem 1.1. It applies to two similar truncated second order cones: if one of them does not contain a point of a lattice, then the lattice points contained in the other truncated cone lie on a number of hyperplanes which only depends on n.

Lemma 2.5. *Let Λ be an arbitrary lattice in $\mathbb{R}^n$. Moreover, let*

$$C := \left\{ x \in \mathbb{R}^n \;\middle|\; \frac{1}{2(n-1)} \sum_{i=1}^{n-1} x_i^2 \leq x_n \leq 1 \right\},$$

and let

$$\bar{C} := \left\{ x \in \mathbb{R}^n \;\middle|\; \sum_{i=1}^{n-1} x_i^2 \leq x_n \leq 1 \right\}.$$

If $\operatorname{int}(\bar{C}) \cap \Lambda = \emptyset$, then the lattice points $C \cap \Lambda$ lie on at most $4^n n^{3n}$ hyperplanes.

Proof. Our idea is to cover C with $4^n n^{3n}$ boxes. Then we show that the lattice points in each box lie on a single hyperplane. We note that, if a convex set $L \subset \mathbb{R}^n$ satisfies $L + \Lambda = \mathbb{R}^n$, then any translate of L contains at least one point of Λ. Furthermore, observe that, for any points $v_0, \ldots, v_n \in [0, \frac{1}{n^2}]^n$, it holds that

$$\left\{ x \in \mathbb{R}^n \;\middle|\; x = \sum_{i=1}^{n} \lambda_i (v_i - v_0) \text{ and } -\frac{1}{2} \leq \lambda_i \leq \frac{1}{2} \text{ for all } i = 1, \ldots, n \right\} \subset \left[-\frac{1}{2n}, \frac{1}{2n} \right]^n \tag{1}$$

and for a sufficiently small $\alpha > 0$ it holds

$$\left[-\frac{1}{2n}, \frac{1}{2n} \right]^{n-1} \times \left[\frac{n-1-\alpha}{n}, \frac{n-\alpha}{n} \right] \subset \operatorname{int}(\bar{C}). \tag{2}$$

It is straightforward to check that the right hand side in (1) and the left hand side in (2) are translates. More precisely, the set $[-\frac{1}{2n}, \frac{1}{2n}]^n + \frac{2(n-\alpha)-1}{2n} e_n = [-\frac{1}{2n}, \frac{1}{2n}]^{n-1} \times [\frac{n-1-\alpha}{n}, \frac{n-\alpha}{n}]$, where e_n denotes the n-th unit vector. Observe that $C \subset [-2n, 2n)^n$. Next we partition $[-2n, 2n)^n$ into boxes. Let $D := [-2n, 2n)^n \cap \frac{1}{n^2}\mathbb{Z}^n$. Then the cardinality of D is $4^n n^{3n}$. Moreover, $C \subset [-2n, 2n)^n \subset D + [0, \frac{1}{n^2}]^n$. Now assume that there exists a box $d + [0, \frac{1}{n^2}]^n$, with $d \in D$, that contains $n+1$ affinely independent lattice points $v_0, \ldots, v_n$, i.e. assume that $v_0, \ldots, v_n \in \Lambda \cap (d + [0, \frac{1}{n^2}]^n)$. Then $\{x \in \mathbb{R}^n \mid x = \sum_{i=1}^{n} \lambda_i (v_i - v_0) \text{ and } -\frac{1}{2} \leq \lambda_i \leq \frac{1}{2} \text{ for all } i = 1, \ldots, n\} + \Lambda = \mathbb{R}^n$. This, together with (1) and (2), contradicts $\operatorname{int}(\bar{C}) \cap \Lambda = \emptyset$. ☐

In order to apply Lemma 2.5 in our proof of Theorem 1.1, we will adapt it to the notation that will be used later and we will show that we can compute the hyperplanes efficiently.

Lemma 2.6. *Let Λ be an arbitrary lattice. Let $P \subset \mathbb{R}^n$ be a $(n-1)$-dimensional polytope, and let $\bar{P}$ be defined as in Lemma 2.2. Furthermore, let $y \in \mathbb{R}^n \setminus \operatorname{aff}(P)$ such that $\operatorname{int}\big(\operatorname{conv}(\{y\}, \bar{P})\big) \cap \Lambda = \emptyset$. In polynomial time in the input size of P and y, we can construct hyperplanes containing all the lattice points $\operatorname{conv}(\{y\}, P) \cap \Lambda$. The number of hyperplanes is at most $4^n n^{3n}$.*

Proof. From Lemma 2.2, it follows that there exists a $(n-1)$-dimensional ellipsoid E and a point $c \in \bar{P}$ such that $c + E \subset \bar{P}$ and $P \subset c + 2(n-1)E$. In particular we can compute $\bar{P}$ in polynomial time (see Remark 2.3) and hence we can compute E and c in polynomial time (see [4, Theorem 3.3.3] or [12]). Moreover, there exists a bijective affine mapping $A : \mathbb{R}^n \mapsto \mathbb{R}^n$ such that $A\left(\operatorname{conv}\left(\{y\}, c + 2(n-1)E\right)\right) = C$ and $A\left(\operatorname{conv}\left(\{y\}, c + E\right)\right) = \bar{C}$, with C and $\bar{C}$ as in Lemma 2.5. By applying Lemma 2.5 to C, $\bar{C}$, and $\Lambda' = A(\Lambda)$ it follows that we can place the lattice points $\operatorname{conv}(\{y\}, P) \cap \Lambda$ onto at most $4^n n^{3n}$ hyperplanes.

It remains to construct the hyperplanes. For that, we use again the notation of the previous Lemma 2.5 and its proof. For every $d \in D$ there exists a hyperplane H_d such that $A^{-1}(d + [0, \frac{1}{n^2}]^n) \cap \Lambda \subset H_d$. By [4, Lemma 6.5.3], we can determine H_d explicitly in polynomial time. ☐

3 Proof of Theorem 1.1.

Let us first outline the main steps of the proof.

We start by applying the Δ-Feasibility Oracle to the polytope $[-B, B]^n$. Assume that the oracle returns a point $y \in K_{\frac{\varepsilon}{2}}$. We then consider an arbitrary facet F of $[-B, B]^n$, and define the set $T_0 := \operatorname{conv}(\{y\}, F)$. The basic idea is to successively construct subsets $T_0 \supset T_1 \supset T_2 \ \ldots$ that satisfy $\operatorname{vol}(T_{i+1}) < \operatorname{vol}(T_i)$ for all i. This subset construction is iterated until we either obtain a set T_i in which we can find an integer point $z \in K_\varepsilon$; or the volume of one of the constructed sets is so small that we can apply Lemma 2.1 to reduce the n-dimensional problem to a $(n-1)$-dimensional problem. By our hypothesis of induction, the $(n-1)$-dimensional problem can be solved in polynomial time.

Let us now explain how the construction of the sets T_i is implemented. Each set T_{i+1} arises from the set T_i by intersecting T_i with a half-space as follows. We first define a certain scaling of F, say $\bar{F}$, such that $\bar{F} \subset F$. Next we employ the Linear Integer Optimization Oracle. If $\operatorname{conv}(\{y\}, \bar{F}) \cap \mathbb{Z}^n = \emptyset$, then Lemma 2.6 implies that we either find a point $z \in T_i \cap K_\varepsilon \cap \mathbb{Z}^n$ by solving a constant number of lower-dimensional problems, or we know that $T_i \cap K_0 \cap \mathbb{Z}^n = \emptyset$. On the other hand, if $\operatorname{conv}(\{y\}, \bar{F}) \cap \mathbb{Z}^n \neq \emptyset$, then we compute a point $x^\star \in \operatorname{conv}(\{y\}, \bar{F}) \cap \mathbb{Z}^n$ closest to y with respect to the normal vector of $\operatorname{aff}(F)$. Let $H^\star$ be the hyperplane parallel to $\operatorname{aff}(F)$ and passing through $x^\star$. Then we use the Separating Hyperplane Oracle to determine a $(n-2)$-dimensional hyperplane $S^\star$ in $H^\star$ separating $x^\star$ from the level set $H^\star \cap K_{\frac{\varepsilon}{2}}$. In turn, $S^\star$ is lifted to the $(n-1)$-dimensional hyperplane $S := \operatorname{aff}(\{y\}, S^\star)$. Let S^+ be the half-space containing $H^\star \cap K_{\frac{\varepsilon}{2}}$ and having S as its boundary. We then define $T_{i+1} := T_i \cap S^+$. Lemma 2.4 guarantees a sufficient decrease of the volume of T_{i+1} with respect to T_i. It remains to check the integer points between $H^\star$ and the hyperplane parallel to $H^\star$ and passing through y. For this, we employ Lemma 2.6 again.

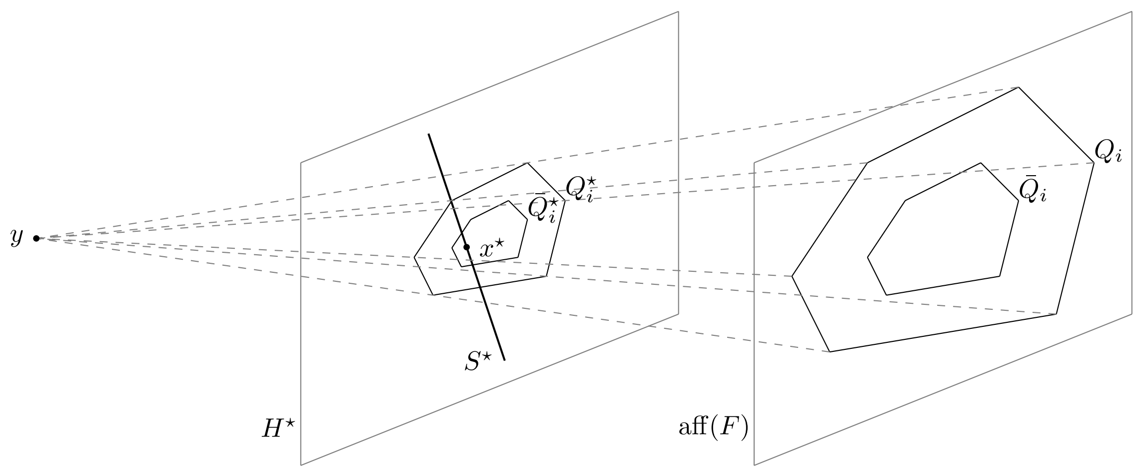

Figure 1: Construction of the truncated cones in the proof of Theorem 1.1.

Proof of Theorem 1.1. First, we apply the Δ-Feasibility Oracle to $P = [-B, B]^n$. If the oracle returns that there is no point in K_0, then $K_0 \cap \mathbb{Z}^n = \emptyset$. So let us assume that the oracle returns a point $y \in K_{\frac{\varepsilon}{2}}$.

We use induction on the dimension n. If $n = 1$, then we just check whether $\lfloor y \rfloor \in K_\varepsilon$ or $\lceil y \rceil \in K_\varepsilon$. In the following let us assume that $n \geq 2$, and that we can solve all lower-dimensional problems. Let $F_1, \ldots, F_{2n}$ be the facets of $[-B, B]^n$. Then

$$[-B, B]^n = \bigcup_{j=1}^{2n} \operatorname{conv}(\{y\}, F_j).$$

The following procedure is applied to every facet of $[-B, B]^n$. Hence let us only consider an arbitrary facet $F \in \{F_1, \ldots, F_{2n}\}$. We define

$$Q_0 := F \quad \text{and} \quad T_0 := \operatorname{conv}(\{y\}, Q_0).$$

Let σ denote the Euclidean distance of y to $\operatorname{aff}(F)$. Then the volume of T_0 is proportional to the $(n-1)$-dimensional volume of Q_0 times σ. More precisely, $\operatorname{vol}(T_0) = \frac{\sigma}{n} \operatorname{vol}_{n-1}(Q_0)$. In the following, we construct

a sequence $T_0 \supset T_1 \supset T_2 \ \ldots$. The construction terminates either with a set T_k which contains an integer point $z \in K_\varepsilon$ we can find; or with the conclusion that no integer point in $T_0 \cap K_0$ exists. Below we show that either way we need to perform at most $O(\log(B))$ steps. Moreover, we show that the iterative construction of a set T_{i+1} from T_i is performed in polynomial time, and that $(T_0 \setminus T_i) \cap K_0 \cap \mathbb{Z}^n = \emptyset$, and $\operatorname{vol}(T_{i+1}) \leq \left(1 - 2^{-n}(n-1)^{1-n}\right) \operatorname{vol}(T_i)$ for all i. Since in each step the decrease of the volume is only dependent on n, it is guaranteed that for some $k \in O(\log(B))$ it holds that $\operatorname{vol}(T_k) < \frac{1}{n!}$. Then, due to Lemma 2.1, it follows that $\dim(T_k \cap \mathbb{Z}^n) \leq n-1$, and we can easily determine whether $T_k \cap K_0 \cap \mathbb{Z}^n$ is empty or find a point in $T_k \cap K_\varepsilon \cap \mathbb{Z}^n$, by induction.

The iterative construction is as follows. Let Q_i and T_i be given. First we define the auxiliary polytopes

$$\bar{Q}_i := \left\{ x \in \mathbb{R}^n \;\middle|\; x + \frac{1}{4n}(Q_i - Q_i) \subset Q_i \right\} \quad \text{and} \quad \bar{T}_i := \operatorname{conv}(\{y\}, \bar{Q}_i)$$

(see Figure 1 and Remark 2.3). Next we employ the Linear Integer Optimization Oracle to solve the linear integer program

$$\min h^\mathsf{T} x \text{ s.t. } x \in \bar{T}_i \cap \mathbb{Z}^n, \tag{3}$$

where h is the normal vector of $\operatorname{aff}(F)$ such that $h^\mathsf{T} y < h^\mathsf{T} x$ for $x \in F$. We distinguish two cases.

Case 1 The linear integer program (3) is infeasible. Then $\bar{T}_i \cap \mathbb{Z}^n = \emptyset$. By construction, we can apply Lemma 2.6 to determine whether there exists an $z \in (T_i \setminus \bar{T}_i) \cap K_\varepsilon \cap \mathbb{Z}^n$ or whether $(T_i \setminus \bar{T}_i) \cap K_0 \cap \mathbb{Z}^n = \emptyset$. This requires to solve at most $k \leq 4^n n^{3n}$ subproblems of dimension $n-1$. Let these subproblems be contained in the hyperplanes $H_1, \ldots, H_k$. Then, for all $j = 1, \ldots, k$, we test the lower-dimensional sets

$$T_i \cap H_j \cap K_0 \cap \mathbb{Z}^n$$

for feasibility. By assumption of induction, all these problems can be solved in polynomial time.

Case 2 The linear integer program (3) has an optimal solution $x^\star$. If $x^\star \in K_\varepsilon$, then we are done. Otherwise, let $H^\star := \{x \in \mathbb{R}^n \mid h^\mathsf{T} x = h^\mathsf{T} x^\star\}$, i.e. the hyperplane containing $x^\star$ and being parallel to $\operatorname{aff}(F)$. We define

$$\begin{aligned} Q_i^\star &:= T_i \cap H^\star \quad \text{and} \quad T_i^\star := \operatorname{conv}(\{y\}, Q_i^\star), \\ \bar{Q}_i^\star &:= \bar{T}_i \cap H^\star \quad \text{and} \quad \bar{T}_i^\star := \operatorname{conv}(\{y\}, \bar{Q}_i^\star). \end{aligned}$$

Using the Separating Hyperplane Oracle with $A = H^\star$ and $a = x^\star$, let $S^\star \subset H^\star$ be a $(n-2)$-dimensional hyperplane containing $x^\star$ and separating $x^\star$ from $H^\star \cap K_{\frac{\varepsilon}{2}}$. Next let S denote the unique $(n-1)$-dimensional hyperplane containing y and $S^\star$, i.e. $S := \operatorname{aff}(\{y\}, S^\star)$. Furthermore, let S^+ denote the half-space with boundary S, and containing $H^\star \cap K_{\frac{\varepsilon}{2}}$. Then, due to the convexity of the level set, we observe

$$\left(\left((T_i \setminus T_i^\star) \setminus H^+\right) \cap K_0\right) \subset \left(\left((T_i \setminus T_i^\star) \setminus H^+\right) \cap K_{\frac{\varepsilon}{2}}\right) = \emptyset. \tag{4}$$

It remains to check for an improving integer point $z \in K_\varepsilon$ within $T_i^\star \setminus \bar{T}_i^\star$. For that, we apply Lemma 2.6 to $T_i^\star$ and $\bar{T}_i^\star$ in the same way that we described in Case 1. If none of the corresponding subproblems returns a point $z \in K_\varepsilon \cap \mathbb{Z}^n$, then together with (4) we know that $(T_i \setminus S^+) \cap K_0 \cap \mathbb{Z}^n = \emptyset$. We define

$$Q_{i+1} := Q_i \cap S^+ \quad \text{and} \quad T_{i+1} := T_i \cap S^+.$$

It holds that $z \notin K_0$ for all $z \in (T_0 \setminus T_{i+1}) \cap \mathbb{Z}^n$. In particular, from Lemma 2.4 it follows, that $\operatorname{vol}_{n-1}(Q_{i+1}) \leq (1 - 2^{-n}(n-1)^{1-n}) \operatorname{vol}_{n-1}(Q_{i+1})$. Hence, $\operatorname{vol}(T_{i+1}) \leq (1 - 2^{-n}(n-1)^{1-n}) \operatorname{vol}(T_i)$. □

4 Extension to the mixed integer setting.

It is straightforward to extend Theorem 1.1 to the mixed integer setting with a constant number of integer variables $z_1, \ldots, z_n$ and any number of continuous variables $x_1, \ldots, x_d$. Simply replace any query with input $z \in \mathbb{Z}^n$ to the evaluation oracle $f(\cdot)$ by a call of a Δ-Feasibility Oracle applied to a fixed $z^\star \in \mathbb{Z}^n$ and returning the value $\min\{f(z^\star, x) \mid (z^\star, x) \in P\}$.

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